Cardy's Formula for some Dependent Percolation Models

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Abstract

We prove Cardy's formula for rectangular crossing probabilities in *dependent* site percolation models that arise from a deterministic cellular automaton with a random initial state. The cellular automaton corresponds to the zero-temperature case of Domany's stochastic Ising ferromagnet on the hexagonal lattice $\mathbb{H}$ (with alternating updates of two sublattices) [7]; it may also be realized on the triangular lattice $\mathbb{T}$ with flips when a site disagrees with six, five and sometimes four of its six neighbors.

1 Introduction

It was understood by physicists since the early seventies that critical statistical mechanics models should possess continuum scaling limits with a global conformal invariance that goes beyond pure scale invariance. The phenomenon is particularly interesting in two dimensions, where every analytic function gives rise to a conformal transformation and the local conformal transformations form an infinite dimensional group; in that context, it was first studied by Belavin, Polyakov and Zamolodchikov [1, 2]. For an introduction to the methods of conformal field theory as applied to two-dimensional critical percolation, see [6].

Until recently, however, there was no rigorous mathematical proof of this phenomenon, with the exception of the Simple Symmetric Random Walk, whose continuum scaling limit

is Brownian Motion. Then, S. Smirnov managed to prove [20, 21] existence, uniqueness and conformal invariance of the continuum scaling limit of critical site percolation on the triangular lattice, obtaining in particular conformal invariance of crossing probabilities and Cardy's formula for rectangular crossings [5, 6].

In this paper we show that there are some natural *dependent* percolation models for which conformal invariance of the crossing probabilities and Cardy's formula can be proved. Our proof relies on Smirnov's result and on properties of the dependent percolation models which make them, in a sense to be specified later, "small perturbations" of the independent model treated by Smirnov.

The dependent percolation models we consider are the distributions at time $n \geq 1$ (including the final state as $n \to \infty$) of a discrete time deterministic dynamical process σ^n with state space $\{-1,+1\}^{\mathbb{L}}$ consisting of assignments of -1 or $+1$ to a regular lattice $\mathbb{L}$. The initial σ^0 is "uniformly random", i.e., the distribution of σ^0 is a Bernoulli(1/2) product measure. The dynamics are those of Domany's stochastic Ising ferromagnet [7] at zero temperature. There are two essentially equivalent versions — one where $\mathbb{L}$ is the hexagonal lattice $\mathbb{H}$ and one where it is the triangular lattice $\mathbb{T}$. We take $\mathbb{H}$ and $\mathbb{T}$ to be regular lattices embedded in $\mathbb{R}^2$ so that the elementary cells of $\mathbb{H}$ (resp., $\mathbb{T}$) are regular hexagons (resp., equilateral triangles). In the first version, $\mathbb{H}$, as a bipartite graph, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$ which are alternately updated so that each σ_x is forced to agree with a majority of its three neighbors (which are in the other subset). In the second version, all sites are updated simultaneously according to a rule based on a deterministic pairing of the six neighbors of every site into three pairs (see the end of Section 2 for a complete explanation). The rule is that σ_x flips if and only if it disagrees (after the previous update) with both sites in two or more of its three neighbor pairs; thus there is (resp., is not) a flip if the number D_x of disagreeing neighbors is ≥ 5 (resp., ≤ 3) and there is also a flip for some cases of $D_x = 4$. We note that Cardy's formula can also be verified for a modified rule in which there is a flip if and only if $D_x \geq 5$; the case of a modified rule where there is a flip if and only if $D_x \geq 4$ is an interesting open problem.

2 Definition of the model(s) and results

In this section we give a more detailed description of the dependent percolation models and results.

Consider the homogeneous ferromagnet on the hexagonal lattice $\mathbb{H}$ with states denoted by $\sigma = \{\sigma_x\}_{x\in\mathbb{H}}$, $\sigma_x = \pm 1$, and with (formal) Hamiltonian

$$\mathcal{H} = -\sum_{\langle x,y\rangle} \sigma_x \sigma_y, \tag{1}$$

where $\sum_{\langle x,y\rangle}$ denotes the sum over all pairs of neighbor sites, each pair counted once. The variables σ_x, σ_y are called spins. We write $\mathcal{N}^{\mathbb{H}}(x)$ for the set of three neighbors of x, and

indicate with

$$\Delta_x \mathcal{H}(\sigma) = 2 \sum_{y \in \mathcal{N}^{\mathbb{H}}(x)} \sigma_x \sigma_y \tag{2}$$

the change in the Hamiltonian when the spin σ_x at site x is flipped (i.e., changes sign).

Notice that the hexagonal lattice can be partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$ in such a way that all three neighbors of any site in $\mathcal{A}$ (resp., $\mathcal{B}$) are in $\mathcal{B}$ (resp., $\mathcal{A}$). By placing an edge between any two sites of $\mathcal{A}$ (resp., $\mathcal{B}$) that are next-nearest neighbors in $\mathbb{H}$, the subset $\mathcal{A}$ (resp., $\mathcal{B}$) becomes a triangular lattice. (This relation between an hexagonal lattice and its triangular "sublattice," sometimes expressed in terms of a "star-triangle transformation," will be used again in Remark 2.1 below.) We now consider the discrete time Markov process σ^n, $n \in \mathbb{N}$, with state space $\mathcal{S} = \{-1, +1\}^{\mathbb{H}}$, which is the zero temperature limit of a model of Domany [7], constructed as follows:

- The initial state σ^0 is chosen from a symmetric Bernoulli product measure.
- At odd times $n = 1, 3, \ldots$, the spins in the sublattice $\mathcal{A}$ are updated according to the following rule: σ_x, $x \in \mathcal{A}$, is flipped if and only if $\Delta_x \mathcal{H}(\sigma) < 0$.
- At even times $n = 2, 4, \ldots$, the spins in the sublattice $\mathcal{B}$ are updated according to the same rule as for those of the sublattice $\mathcal{A}$.

In order to present the main result of this paper, let us denote by σ^∞ the final state of the process σ^n defined above. $\sigma^\infty = \lim_{n\to\infty} \sigma^n$ exists with probability one, as was proved in [15], and, like σ^n for $1 \leq n < \infty$, defines a dependent percolation model on $\mathbb{H}$. These are the the main objects of our investigation.

We will call δ the "mesh" of the lattice and consider the continuum scaling limit of the dependent percolation model σ^n on $\delta\mathbb{H}$ as $\delta \to 0$. For simplicity of exposition, we will prove Cardy's formula in the special case of a rectangle, aligned with the coordinate axes and of given cross-ratio η (a similar approach would work for any domain with a "regular" boundary, but it would involve dealing with more complex deformations of the boundary). Consider a finite rectangle $\mathcal{R} = \mathcal{R}(a, b) \equiv (-a/2, a/2) \times (-b/2, b/2) \subset \mathbb{R}^2$ with sides of lengths a and b, such that the cross-ratio a/b is η. We say that there is (in σ^n) a vertical plus-crossing if $\mathcal{R} \cap \delta\mathbb{H}$ contains a path of $+1$ spins from σ^n joining the top and bottom sides of the rectangle $\mathcal{R}$, and call $P_\delta(\eta; n)$ the probability of such a plus-crossing at time n. More precisely, there is a vertical plus crossing if there is a path $x_0, x_1, \ldots, x_m, x_{m+1}$ in $\mathbb{H}$ with $\sigma^n_{x_j} = +1$ for all j, with $\delta x_1, \ldots, \delta x_m$ all in $\mathcal{R}$, and with the line segments $\overline{\delta x_0, \delta x_1}$ and $\overline{\delta x_m, \delta x_{m+1}}$ touching respectively the top side $[-a/2, a/2] \times \{b/2\}$ and the bottom side $[-a/2, a/2] \times \{-b/2\}$. In the next section we will prove the following result:

Theorem 1. *For all $n \geq 1$ (including $n = \infty$), the limit $P(\eta; n) = \lim_{\delta\to 0} P_\delta(\eta; n)$ exists and is given by Cardy's formula:*

$$P(\eta; n) = F_C(\eta) \equiv \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} \, \eta^{\frac{1}{3}} \, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \eta\right). \tag{3}$$

A stronger result than Theorem 1 can be obtained, i.e., it is possible to prove existence, uniqueness and conformal invariance of the continuum scaling limit, as proven by Smirnov [20, 21] for independent site percolation on the triangular lattice. Such a result, though, requires more work and will be pursued in a future paper. Here we just note that the proof is based on showing that the limit for our dependent percolation models (on the hexagonal lattice) coincides with that of Smirnov for independent percolation on the triangular lattice, i.e., that the models belong to the same universality class.

The following observations are useful in understanding the behavior of the model and will help in the proof of Theorem 1.

- The values of the spins in the sublattice $\mathcal{A}$ at time zero are irrelevant, since at time 1, after the first update, those values are uniquely determined by the values of the spins in the sublattice $\mathcal{B}$.
- Once the initial spin configuration in the sublattice $\mathcal{B}$ is chosen, the dynamics is completely deterministic.
- A site can no longer flip once it belongs to either a loop or "barbell" of constant sign in $\mathbb{H}$, where a loop means a simple loop (with no subloops) and a barbell consists of two disjoint loops connected by a path (we regard a loop as a degenerate barbell).

We also note that, by studying the percolation properties of the final state σ^∞ on the infinite lattice $\mathbb{H}$, it can be shown that every site is in some barbell of constant σ^∞-sign [4].

The discrete time Markov process defined above can be considered a simplified version of a continuous time process where an independent (rate 1) Poisson clock is assigned to each site $x \in \mathbb{H}$, and the spin at site x is updated (with the same rule as in our discrete time process) when the corresponding clock rings. The percolation properties of the final state σ^∞ of that process were studied, both rigorously and numerically, in [13]; the results there (about critical exponents rather than critical crossing probabilities) strongly suggest that that dependent percolation model is also in the same universality class as independent percolation. Similar stochastic processes on different types of lattices have been studied in various papers. See, for example, [3, 8, 10, 15, 16, 17, 18] for models on $\mathbb{Z}^d$ and [12] for a model on the homogeneous tree of degree three. Such models are also discussed extensively in the physics literature, usually on $\mathbb{Z}^d$ (see, for example, [7] and [14]). On the hexagonal lattice, the discrete time dynamics is the zero-temperature case of Domany's dynamics [7]. Numerical simulations have been done by Nienhius [19] and rigorous results for both the continuous and discrete dynamics have been obtained in [4], including a detailed analysis of the discrete time (synchronous) case. The analysis of [4] is at the heart of this paper, and we will refer to and heavily rely on it for the proof of Theorem 1, which is given in the next section.

There is an alternative, but equivalent, way of describing the discrete time dynamics as a deterministic cellular automaton on the triangular lattice $\mathbb{T}$ (with random initial state). The initial state is again chosen by assigning value $+1$ or -1 independently, with equal probability, to each site of the triangular lattice. Given some site $\bar{x} \in \mathbb{T}$, group its

six $\mathbb{T}$-neighbors y in three disjoint pairs $\{y_1^{\bar{x}}, y_2^{\bar{x}}\}, \{y_3^{\bar{x}}, y_4^{\bar{x}}\}, \{y_5^{\bar{x}}, y_6^{\bar{x}}\}$, so that $y_1^{\bar{x}}$ and $y_2^{\bar{x}}$ are $\mathbb{T}$-neighbors, and so on for the other two pairs. Translate this construction to all sites $x \in \mathbb{T}$, thus producing three pairs of sites $\{y_1^x, y_2^x\}, \{y_3^x, y_4^x\}, \{y_5^x, y_6^x\}$ associated to each site $x \in \mathbb{T}$. (Note that this construction does not need to specify how $\mathbb{T}$ is embedded in $\mathbb{R}^2$.) Site x is updated at times $m = 1, 2, \ldots$ according to the following rule: the spin at site x is changed from σ_x to $-\sigma_x$ if and only if at least two of its pairs of neighbors have the same sign and this sign is $-\sigma_x$.

Remark 2.1. *This dynamics on the triangular lattice $\mathbb{T}$ is equivalent to the alternating sublattice dynamics on the hexagonal lattice $\mathbb{H}$ when restricted to the sublattice $\mathcal{B}$ for even times $n = 2m$. To see this, start with $\mathbb{T}$ and construct an hexagonal lattice $\mathbb{H}'$ by means of a star-triangle transformation (see, for example, p. 335 of [11]) such that a site is added at the center of each of the triangles $(x, y_1^x, y_2^x), (x, y_3^x, y_4^x)$, and (x, y_5^x, y_6^x). $\mathbb{H}'$ may be partitioned into two triangular sublattices $\mathcal{A}'$ and $\mathcal{B}'$ with $\mathcal{B}' = \mathbb{T}$. It is now easy to see that the dynamics on $\mathbb{T}$ for $m = 1, 2, \ldots$ and the alternating sublattice dynamics on $\mathbb{H}'$ restricted to $\mathcal{B}'$ for even times $n = 2m$ are the same.*

Theorem 1 (and its generalizations) in this context means that, at all times $m \geq 0$, the crossing probabilities for the states σ^m of this cellular automaton on $\mathbb{T}$ have the same conformally invariant continuum scaling limit as that for critical independent percolation on $\mathbb{T}$, despite the dependence induced by the cellular automaton dynamics.

3 Proof of Theorem 1

In this final section of the paper we prove Theorem 1. We follow the notation of [4] and start by giving some definitions. Let us consider a loop γ in the triangular sublattice $\mathcal{B}$, written as an ordered sequence of sites $(y_0, y_1, \ldots, y_n)$ with $n \geq 3$, which are distinct except that $y_n = y_0$. For $i = 1, \ldots, n$, let ζ_i be the unique site in $\mathcal{A}$ that is an $\mathbb{H}$-neighbor of both y_{i-1} and y_i. We call γ an *s-loop* if $\zeta_1, \ldots, \zeta_n$ are all distinct. Similarly, a (site-self avoiding) path $(y_0, y_1, \ldots, y_n)$ in $\mathcal{B}$, between y_0 and y_n, is called an *s-path* if $\zeta_1, \ldots, \zeta_n$ are all distinct. Notice that any path in $\mathcal{B}$ between y and y' (seen as a collection of sites) contains an s-path between y and y'. An s-loop of constant sign is stable for the dynamics since at the next update of $\mathcal{A}$ the presence of the constant sign s-loop in $\mathcal{B}$ will produce a stable loop of that sign in the hexagonal lattice. Similarly an s-path of constant sign between y and y' will be stable if y and y' are stable — e.g., if they each belong to an s-loop. A triangular loop $x_1, x_2, x_3 \in \mathcal{B}$ with a common $\mathbb{H}$-neighbor $\zeta \in \mathcal{A}$ is called a *star*; it is not an s-loop. A triangular loop in $\mathcal{B}$ that is not a star is an s-loop and will be called an *antistar*, while any loop in $\mathcal{B}$ that contains more than three sites contains an s-loop.

Before stating a lemma, that will be a main ingredient in the proof of Theorem 1, we need one more definition. For (x, x') an ordered pair of neighbors in $\mathcal{B}$, we define the "partial cluster" $C^{\mathcal{B}}_{(x,x')}$ to be the set of sites $y \in \mathcal{B}$ such that there is a (site-self avoiding) path $x_0 = x', x_1, \ldots, x_n = y$ in $\mathcal{B}$ of constant sign in σ^0, with $x_1 \neq x$ and

$(x_0 = x', x_1, x)$ not forming a star. Combining the stability properties of s-loops and s-paths just discussed, we have the following lemma.

Lemma 3.1. *An s-path $(y_0, \dots, y_m)$ in $\mathcal{B}$ of constant sign in σ^0 is stable (i.e., retains that same sign in σ^n for all $0 \le n \le \infty$) if $C^{\mathcal{B}}_{(y_1,y_0)}$ and $C^{\mathcal{B}}_{(y_{m-1},y_m)}$ both contain s-loops.*

Proof of Lemma 3.1. The original s-path $(y_0, \dots, y_m)$ is stable because either y_0 and y_m both belong to s-loops of constant sign in σ^0 or else there is a longer s-path of constant sign in σ^0, between some y and y' (with the original $(y_0, \dots, y_m)$ as a subpath), such that both y and y' belong to s-loops of constant sign in σ^0. $\square$

With this preparation, we are now ready to start the proof of Theorem 1. What we will prove, roughly speaking, is that, in the limit $\delta \to 0$, there exists a vertical plus-crossing of $\mathcal{R}$ from σ^n with $n \ge 1$, in $\mathcal{R} \cap \delta\mathbb{H}$, if and only if there exists a vertical plus-crossing of $\mathcal{R}$ from σ^0 in $\mathcal{R} \cap \delta\mathcal{B}$. Since $\mathcal{B}$ is a triangular lattice and the initial state σ^0 is chosen from a symmetric Bernoulli product measure, this implies that the limit $P(\eta; n) = \lim_{\delta\to 0} P_\delta(\eta; n)$ exists for $n \ge 1$ and is the same as in the case of the crossing probability for independent site percolation on the triangular lattice, thus proving the theorem.

Consider two rectangles, $\mathcal{R}' = \mathcal{R}(a', b')$ with b' slightly larger than b and a' slightly smaller than a, and $\mathcal{R}'' = \mathcal{R}(a'', b'')$ with b'' slightly smaller than b and a'' slightly larger than a. Call $P'_\delta(a', b')$ the probability of a vertical plus-crossing from σ^0 in $\mathcal{R}' \cap \delta\mathcal{B}$ joining the top and bottom sides of $\mathcal{R}'$ and $P''_\delta(a'', b'')$ the probability of a *horizontal minus-*crossing from σ^0 in $\mathcal{R}'' \cap \delta\mathcal{B}$ joining the left and right sides of $\mathcal{R}''$. Note that a vertical plus crossing (on the triangular lattice $\delta\mathcal{B}$) occurs if and only if a horizontal minus-crossing does not occur. Clearly, from [20, 21] we have

$$P'(a', b') \equiv \lim_{\delta\to 0} P'_\delta(a', b') = F_C(a'/b'), \tag{4}$$

$$\lim_{a'\to a,\, b'\to b} P'(a', b') = P'(a, b) = F_C(\eta), \tag{5}$$

and

$$\lim_{a''\to a,\, b''\to b}\ \lim_{\delta\to 0} P''_\delta(a'', b'') = 1 - F_C(\eta). \tag{6}$$

Any vertical plus-crossing of $\mathcal{R}' \cap \delta\mathcal{B}$ at time 0 yields a vertical plus-crossing by some s-path $(y_0, \dots, y_m)$, which then yields at time 1 a vertical plus-crossing of $\mathcal{R} \cap \delta\mathbb{H}$ by a path $(y_{k_1}, \zeta_{k_1+1}, \dots, \zeta_{k_2}, y_{k_2})$, providing $a' < a$, $b' \ge b$ and δ is sufficiently small. (The reason we first take $b' > b$ and then let $b' \to b$ is to handle the case of time $n > 1$, as we shall see.) Therefore, for small δ,

$$P_\delta(\eta; n = 1) \ge P'_\delta(a', b'). \tag{7}$$

On the other hand, if there is a horizontal minus-crossing of $\mathcal{R}'' \cap \delta\mathcal{B}$ at time 0, it produces a horizontal minus-crossing in $\mathcal{R} \cap \delta\mathbb{H}$ at time 1 (for small δ) which blocks any possible vertical plus-crossing in $\mathcal{R} \cap \delta\mathbb{H}$ at that time; therefore, for small δ,

$$P_\delta(\eta; n = 1) \leq 1 - P''_\delta(a'', b''). \tag{8}$$

Letting $\delta \to 0$ and then $a', a'' \to a$ and $b', b'' \to b$ and using (5)-(8), we conclude that $P_\delta(\eta; n = 1)$ converges to Cardy's formula, $F_C(\eta)$, as $\delta \to 0$.

It remains to prove that the same is true for all times $n \geq 2$. In order to do that, we first have to show that our vertical plus-crossing of $\mathcal{R}' \cap \delta\mathbb{H}$ by $(y_0, \zeta_1, \ldots, \zeta_m, y_m)$ created at time 1 doesn't "shrink" too much due to the effect of the dynamics, so that at all later times, including $n = \infty$, there is a vertical plus-crossing of $\mathcal{R} \cap \delta\mathbb{H}$ by $(y_{k_1}, \zeta_{k_1+1}, \ldots, \zeta_{k_2}, y_{k_2})$.

To do this by extending the bound (7) to all $n \geq 1$, at the cost of a correction to the right hand side that tends to zero with δ, we apply Lemma 3.1. Noting that each of the partial paths $(y_0, \ldots, y_{k_1})$ and $(y_{k_2}, \ldots, y_m)$ contains of the order of $(b' - b)/\delta$ sites, we see that the lemma implies that it suffices to show that there is some $\beta > 0$ and $K < \infty$ such that for any deterministic (x, x'),

$$P(|C^{\mathcal{B}}_{(x,x')}| \geq \ell \text{ and } C^{\mathcal{B}}_{(x,x')} \text{ contains no antistar}) \leq K\, e^{-\beta \ell}. \tag{9}$$

To prove (9), we partition $\mathcal{B}$ into disjoint antistars and denote by τ the collection of these antistars. We do an algorithmic construction of $C^{\mathcal{B}}_{(x,x')}$ (as in, e.g., [9]), where the order of checking the sign of sites is such that when the first site in an antistar from τ is checked (and found to have the same sign as x'), then the other two sites in that antistar are checked next. Without loss of generality, we assume that $\sigma^0_{x'} = +1$. Then standard arguments show that the probability in (9) is bounded by $K\,(1 - (\frac{1}{2})^3)^{(\ell/3)}$.

To similarly extend the bound (8), one proceeds in the same way, but considering horizontal minus-crossings of $\mathcal{R}'' \cap \delta\mathcal{B}$ at time zero which produce horizontal minus-crossings of $\mathcal{R} \cap \delta\mathbb{H}$ at time $n \geq 1$. Taking the limits $\delta \to 0$, $a' \to a$, $b' \to b$ concludes the proof. $\square$

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